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Covariant Derivation of the Electromagnetic Boundary Conditions in General Relativity

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Abstract

The covariant derivation of the electromagnetic boundary conditions seems not to have been given. It is presented here, for general electromagnetic sources and distributions of matter which are independent of time. Thus the boundary conditions are suitable for use in stationary solutions in General Relativity.

1. *Geometrical Preliminaries*

In V_4, the manifold of space-time, select a vector t_μ, not null. Select a finite part, V_3 say, of the hypersurface orthogonal to t_μ, and call the surface of V_3, V_2. The V_3 and V_2 must be chosen as continuous piecewise diffeomorphic images of B^3 and S^2, the closed unit Euclidean ball of three dimensions and its surface, respectively. Let V_2 have a normal n_σ in the hypersurface orthogonal to t_μ, so that $t_\mu n^\mu = 0$. By suitable choice of V_3 and V_2, n_μ is to be made not null anywhere on V_2; in particular, n_μ may make discontinuous changes from space-like to time-like and back. In the closed V_2, select a closed V_1, an image as above of S^1, the unit circle, with a tangent L_μ. V_1 must be chosen such that L_μ is not null, but L_μ may change discontinuously from space-like to time-like and back. Let W_2 be any open two surface, an image as above of E^2, the open unit disc, such that $W_2 \cap V_2 = V_1$, and that W_2 has a non-null normal N_μ in V_3. N_μ must be either time-like or space-like all over W_2, and must not change from one to the other. N_μ is to lie in V_3; thus $N_\mu t^\mu = 0$. All these vectors are units. W_2 (respectively V_1) divides V_3 (V_2) into two regions, E and I ($\mathscr{E}$ and $\mathscr{I}$) say. The letters E, I, $\mathscr{E}$, $\mathscr{I}$, W_2, are used to denote functions defined in that region (see Fig. 1).

These regions may be regarded as successive embeddings. Use x^α, x^A, x^a

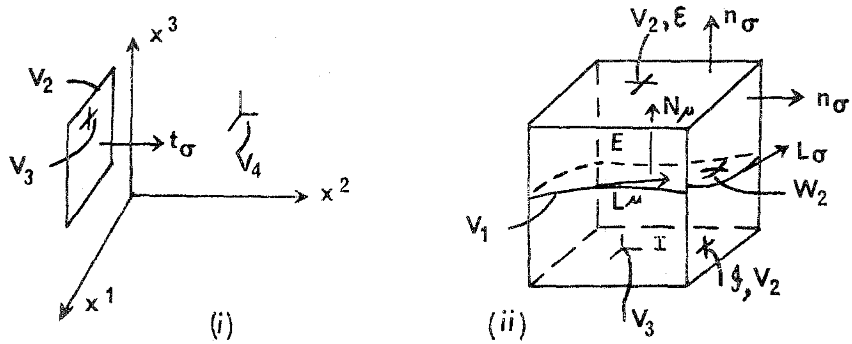

Figure 1.—(i) a V_3 in V_4, (ii) a V_3.

($\alpha = 1, 2, 3, 4$; $A = 1, 2, 3$; $a = 1, 2$) to denote coordinates in general V_4, V_3, V_2, respectively. Then for example

$$E = \{x^\alpha : x^\alpha = \underset{E}{f^\alpha}(x^A)\}$$

$$\mathscr{E} = \{x^A : x^A = \underset{\mathscr{E}}{g^A}(x^a)\}$$

In particular, W_2 is regarded as being embedded in either E or I:

$$W_2 = \{x^A : x^A = \underset{E}{h^A}(x^a)\}$$

$$= \{x^A : x^A = \underset{I}{h^A}(x^a)\}$$

Correspondingly, a scalar $\phi(x^\alpha)$, defined in a region of V_4 which includes for example E, has successively induced values

$$\underset{E}{\phi} = \phi(x^\alpha = \underset{E}{f^\alpha}(x^A))$$

$$\underset{\mathscr{E}}{\phi} = \underset{E}{\phi}(x^A = \underset{\mathscr{E}}{g^A}(x^a))$$

and similarly for I. A scalar $\underset{E}{\phi}$ induces $(\underset{E}{\phi})_{W_2}$ on W_2:

$$(\underset{E}{\phi})_{W_2} = \underset{E}{\phi}(x^A = \underset{E}{h^A}(x^a))$$

and similarly for I. Define $[\phi]$ as

$$[\phi] = (\underset{E}{\phi})_{W_2} - (\underset{I}{\phi})_{W_2}$$

Induced scalars are assumed to be the continuously attained limits of the inducing scalars. In Section 2, it is assumed that the scalars in the integrands are the induced scalars in the appropriate region.

Define two limiting processes, $\lim_{E}$ and $\lim_{I}$, such that for example under $\lim_{E}$,

$$E \to 0, \qquad \underset{E}{n_\mu} \to N_\mu, \qquad \mathscr{E} \to W_2, \qquad \underset{E}{\phi} \to \underset{E}{(\phi)}_{W_2}$$

$\lim_{I}$ is similar except that under $\lim_{I}$ $\underset{I}{n_\mu} \to -N_\mu$ by choice of N_μ as pointing into E.

Finally, given a hypersurface B in V_4:

$$B = \{x^\alpha : x^\alpha = x^\alpha(x^A)\}$$

define tangents and unit tangents respectively by

$$\tau_A^\mu = \frac{\partial x^\mu}{\partial x^A}$$

$$t_A^\mu = (g_{\rho\sigma}\,\tau_A^\sigma\,\tau_A^\rho)^{-1/2}\,\tau_A^\mu$$

Then define

$$g_{AB} = g_{\mu\nu}\,t_A^\mu\,t_B^\nu$$

$$g^{AB} \text{ s.t. } g^{AB}g_{BC} = \delta_C^A$$

$$t_\mu^B = g^{BA}\,t_{\mu A}$$

2. *Derivation of the Boundary Conditions*

The integrations used here are those given by Synge (1960). Retardation is not used in the integration of the field equations, so the resulting boundary conditions apply only to time-independent systems. It is assumed that Einstein's Field Equation is to be solved in V_4, for some distributions of matter, charge, current, and fields of polarisation. V_4 may be divided into regions for this purpose. Within the regions, the metric and all material fields are assumed to have sufficient differentiability. Between the regions, there are hypersurfaces of discontinuity of the metric and material fields. The boundary conditions on the electromagnetic fields are derived without any assumptions about these discontinuities, except that they may exist.

The field equations in their most general form are:

$$F_{\mu\nu;\rho}\,\eta^{\mu\nu\rho\sigma} = 0 \tag{2.1}$$

$$H^*{}_{\rho\sigma;\mu}\,\eta^{\mu\nu\rho\sigma} = 4\pi J^\nu \tag{2.2}$$

where $\eta^{\mu\nu\rho\sigma}$ is the alternating tensor and

$$\tfrac{1}{2}H^*{}_{\rho\sigma}\,\eta^{\mu\nu\rho\sigma} = H^{\mu\nu} \tag{2.3}$$

$$F_{\mu\nu} = H_{\mu\nu} + 4\pi M_{\mu\nu} \tag{2.4}$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{2.5}$$

The conventions of Panofsky and Phillips (1962) are used, except that the units are Gaussian. The equations hold throughout each region of V_4.

To find the conditions resulting from (2.1), select any V_3 with corresponding t_μ, V_2, and n_μ. Then

$$\int_{V_3} F_{\mu\nu;\rho}\, t_\sigma\, \eta^{\mu\nu\rho\sigma}\, d^3 v = \oint_{V_2} \epsilon_n F_{\mu\nu} n_\rho\, t_\sigma\, \eta^{\mu\nu\rho\sigma}\, d^2 v$$
$$= 0 \tag{2.6}$$

This holds for any V_2. Selecting W_2 (and so V_1), and applying the limiting processes in E and I, one has from (2.6)

$$\int_{W_2} [F_{\mu\nu} N_\rho\, t_\sigma\, \eta^{\mu\nu\rho\sigma}]\, d^2 v = 0 \tag{2.7}$$

This holds for any part of the selected W_2, so that

$$[F_{\mu\nu} N_\rho\, t_\sigma\, \eta^{\mu\nu\rho\sigma}] = 0 \tag{2.8}$$

To write this in terms of the potentials, use (2.5) in (2.6):

$$\oint_{V_2} \epsilon_n A_{\nu,\mu} n_\rho\, t_\sigma\, \eta^{\mu\nu\rho\sigma}\, d^2 v = 0 \tag{2.9}$$

Now, V_2 is to be regarded as $\mathscr{E} \cup \mathscr{I}$. The integral theorem then relates the integrals over $\mathscr{E}$ and $\mathscr{I}$ to integrals round V_1, and the integrals round V_1 from $\mathscr{E}$ and $\mathscr{I}$ have opposite signs. Writing dx^μ along V_1 as $dx^\mu = L^\mu d^1 v$, one has from (2.9)

$$\oint_{V_1} [A_\mu L^\mu]\, d^1 v = 0$$

Again this holds for any closed V_1 in W_2, so that

$$[A_\mu L^\mu] = [\Psi_{,\mu} L^\mu] \tag{2.10}$$

for arbitrary scalars $(\underset{E}{\Psi})_{W_2}$, $(\underset{I}{\Psi})_{W_2}$.

Before developing the conditions from (2.2), it is convenient to treat the four-current J^μ under the limiting processes. The interpretation of the treatment is given in Section 3. If $J^\mu(x^\alpha)$ and $t^\mu(x^\alpha)$ are defined in a part of V_4 that contains for example E, then define

$$\underset{E}{J_\nu}\, \underset{E}{t^\nu} = J_\nu t^\nu (x^\alpha = \underset{E}{f^\alpha}(x^A))$$

$$(\underset{E}{I_\nu}\, \underset{E}{t^\nu})_{W_2} = \lim_{W_2 \to 0} \frac{\underset{E}{\operatorname{Lim}}\, \epsilon_N \int_E \underset{E}{J_\nu}\, \underset{E}{t^\nu}\, d^3 v}{\int_{W_2} d^2 v} \tag{2.11}$$

and similarly for I except that $(-\epsilon_N)$ is used. It follows that

$$\int_{W_2} \underset{E}{I_\nu}\, \underset{E}{t^\nu}\, d^2 v = \underset{E}{\text{Lim}}\, \epsilon_N \int_{E} \underset{E}{J_\nu}\, \underset{E}{t^\nu}\, d^3 v \tag{2.12}$$

and similarly for I. The $(-\epsilon_N)$ is used so that the right-hand side of (2.13) below is the algebraic sum of $\underset{E}{I_\nu}\,\underset{E}{t^\nu}$ and $\underset{I}{I_\nu}\,\underset{I}{t^\nu}$, i.e. the net quantity on W_2.

Now, (2.2) gives

$$[H_{\mu\nu} N^\mu t^\nu] = 4\pi [I_\nu t^\nu] \tag{2.13}$$

in the same way as (2.1) gives (2.8). By using (2.4) and (2.5), one obtains the alternative forms of (2.13)

$$\begin{aligned}[F_{\mu\nu} N^\mu t^\nu] &= [(A_{\nu,\mu} - A_{\mu,\nu}) N^\mu t^\nu] \\ &= 4\pi [I_\nu t^\nu + M_{\mu\nu} N^\mu t^\nu]\end{aligned} \tag{2.14}$$

3. *Discussion of the Boundary Conditions*

It is convenient to begin by setting up the relation between the classical geometry and the space-time geometry used here.

Classically, there is a physical boundary $\underset{\sim}{B}$ say, a two surface in space, with normal $\underset{\sim}{N}$ say. The classical boundary conditions are found by taking elements of volume or area, which link the space on either side of $\underset{\sim}{B}$, and collapsing them along $\underset{\sim}{N}$ into elements of area or arc respectively on $\underset{\sim}{B}$. In Section 2, there is a two surface W_2, with two normals N_μ and t_μ, but the limiting process effectively takes place along N_μ.

It is the correspondence between $\underset{\sim}{N}$ and N_μ which provides the connection between the classical and covariant derivations.

In four space, the surface of a material body generates a time-like hypersurface B in V_4, having a space-like normal. This is to be identified with N_μ. Then, in each hypersurface of constant time and in comoving coordinates, $N_\mu = -(\underset{\sim}{N}, 0)$.

Now, for a given space-like N_μ (i.e. for given $\underset{\sim}{N}$), there will be three independent vectors t_μ in V_4 such that $N_\mu t^\mu = 0$. Call these $t_{\mu A}$ ($A = 1, 2, 4$, because one of them, $t_{\mu 4}$, must be time-like). Thus, there will be three independent W_2's for which pairs of N_μ, $t_{\mu A}$, are the normals. They are the three independent W_2's which can be constructed in B, N_μ is the normal to B, and $t_{\mu A}$ are the tangents to B. The $t_{\mu A}$ include L_μ.

Since $t_{\mu 4}$ is time-like, the corresponding W_2 is space-like, and is an element of the physical boundary $\underset{\sim}{B}$. The V_3 associated with $t_{\mu 4}$ is also space-like, and forms the element of volume of classical theory (see Fig. 1(ii)).

The other two of $t_{\mu A}$ are space-like. The W_2 corresponding to each of them contains space-like and time-like directions, since N_μ is space-like, whilst each V_3 is part of the history of a space-like two surface. In these cases, W_2 at some instant forms the classical arc in $\underset{\sim}{B}$ at that instant, whilst

V_2 forms the closed curve linking B. There will be two independent arcs in B corresponding to the two space-like $t_{\mu A}$ (see Fig. 2).

Next, the definition (2.11) of the surface current may be discussed. Under Lim_E, the integral over E becomes an integral over the whole of W_2, and the integrand becomes its induced value on W_2. Thus

$$\text{Lim}_E\, \epsilon_N \int_E J_\nu \underset{E}{t}{}^\nu\, d^3 \underset{E}{v}$$

is the integral over W_2 of the induced current on W_2. To recover the induced current itself, as a function of position on W_2, namely the left-hand side of (2.11), one forms the right-hand side of (2.11) and understands that W_2 goes to zero area about the position in question on W_2. There are three

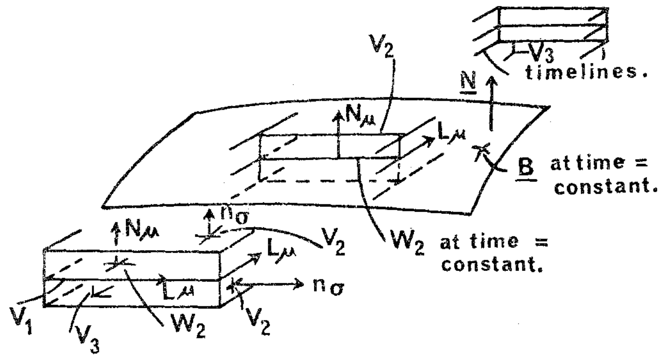

Figure 2.—A V_3 for which t_μ is space-like.

t_μ's in (2.11), for given N_μ, so that, given J_μ in some region of V_4 which contains B, (2.11) defines the surface current vector induced on B:

$$I_A = [I_\nu\, t_A^\nu]$$

where, for each A, the appropriate W_2 is understood. As a vector in V_4, the surface current is given by

$$I_\nu = I_A\, t_\nu^A$$

so that

$$I_\nu N^\nu = 0$$

as is necessary. However, the existence of I_A for a given J^μ and N^μ depends upon there being suitable singularities in J^μ such that under Lim_E, Lim_I, and $\text{Lim}\, W_2 \to 0$, the $(I_\nu t^\nu)_{W_2}$ exist over at least part of W_2. As in the classical case, this will be ensured if a surface current exists and if B is chosen to coincide with the surface in which the current exists.

Now, the boundary conditions may be written as

$$[F_{\mu\nu} N_\rho\, t_{\sigma A}\, \eta^{\mu\nu\rho\sigma}] = 0 \tag{3.1}$$

$$[H_{\mu\nu} N^{\mu} t_A^{\nu}] = 4\pi I_A \tag{3.2}$$

or

$$[F_{\mu\nu} N^{\mu} t_A^{\nu}] = 4\pi I_A + 4\pi [M_{\mu\nu} N^{\mu} t_A^{\nu}] \tag{3.3}$$

and in terms of the potentials

$$[A_{\mu} t_A^{\mu}] = [\Psi_{,\mu} t_A^{\mu}] \tag{3.4}$$

$$[(A_{\nu;\mu} - A_{\mu;\nu}) N^{\mu} t_A^{\nu}] = 4\pi I_A + 4\pi [M_{\mu\nu} N^{\mu} t_A^{\nu}] \tag{3.5}$$

The discontinuity brackets still refer to the two surfaces in V_4. If $t_{\mu 4}$ is used in (3.1) to (3.5), then the classical element of area is intended, but, if $A = 1$, 2, then the classical arcs arise only for time = constant.

As some examples, (3.1) for $A = 4$ gives

$$\mathbf{B} . \underset{\sim}{N} = \text{continuous}$$

and for $A = 1, 2$, gives

$$\mathbf{E} \times \underset{\sim}{N} = \text{continuous}$$

where **B**, **E** are the magnetic induction and electric intensity, respectively, in the limit of flat space-time.

In (3.4), the right-hand side presumably corresponds to double layers of current and charge, by analogy with the classical case. However, (3.4) refers only to the tangential components

$$A_A = A_{\mu} t_A^{\mu}$$

of A_{μ} on B. The normal component (of the discontinuity)

$$[A_N] = [\epsilon_N A_{\mu} N^{\mu}]$$

does not occur, and is set to zero by convention.

In (3.5), sufficient conditions for the terms

$$[A_{\mu;\nu} N^{\mu} t_A^{\nu}] = [A_{N,\nu} t_A^{\nu} - A_{\mu} N^{\mu}{}_{;\nu} t_A^{\nu}]$$

to vanish are that $[A_N] = [A_A] = 0$ and that the first and second induced fundamental forms on B be continuous:

$$\begin{aligned} [h_{AB}] &= [g_{\mu\nu} \tau_A^{\mu} \tau_B^{\nu}] = 0 \\ [k_{AB}] &= [-N_{\mu;\nu} \tau_A^{\mu} \tau_B^{\nu}] = 0 \end{aligned} \tag{3.6}$$

(3.5) then reduces to

$$[A_{\nu;\mu} N^{\mu} t_A^{\nu}] = 4\pi I_A + 4\pi [M_{\mu\nu} N^{\mu} t_A^{\nu}] \tag{3.7}$$

If

$$A_{\nu} = A_N N_{\nu} + A_B t_{\nu}^B$$

is substituted in the left-hand side of (3.7), then

$$[A_{\nu;\mu} N^{\mu} t_A^{\nu}] = [A_{A,N} + A_N N_{\nu;\mu} N^{\mu} t_A^{\nu} + A_B t_{\nu;\mu}^B N^{\mu} t_A^{\nu}] \tag{3.8}$$

where $\phi_{,N} = \phi_{,\nu} N^{\nu}$. The first term in the right-hand side of (3.8) is the classical normal derivative of the tangential components of the potential.

Again, the normal derivative of the normal component does not occur and is made continuous. The other two terms in the right-hand side of (3.8) involve Ricci rotation coefficients of the boundary tetrad $(N_{\mu,}\, t_{\mu A})$. These do not vanish under (3.6) in a general V_4, so that (3.8) reveals a relativistic interaction between the metric and the electromagnetic potentials.

A relativistic effect on the fields is contained in (3.1). If $g = \det|g_{\mu\nu}|$ is discontinuous across B, then (3.1) shows that the invariant components of $F_{\mu\nu}$ on $N_{\mu,}\, t_{\mu A}$, are not by themselves continuous across B. This effect vanishes under (3.6).

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